Effect of anisotropy on finite-size scaling in percolation theory

Mohsen Masihi, Peter R. King, and Peyman Nurafza
Department of Earth Science and Engineering, Imperial College, Exhibition Road, London, SW7 2AZ, United Kingdom

(Received 12 July 2006; published 25 October 2006)

We investigate the effects of anisotropy on finite-size scaling of site percolation in two dimensions. We consider a lattice of size $n_x \times n_y$. We define an aspect ratio $\omega = n_x / n_y$ and consider the mean connected fraction P (averaged over the realizations) as a function of the site occupancy probability (p), the system size (n_x), and this aspect ratio. It is clear that there is an *easy* direction for percolation, which is in the short direction (i.e., y if $\omega > 1$) and a *difficult* direction which is along the long axis. We define an *apparent* percolation threshold in each direction as the value of p when 50% of realizations connect in that direction. We show that standard finite-size scaling[1] applies if we use this apparent threshold. We also find a finite-size scaling for the fluctuations about this mean connected fraction.

DOI: 10.1103/PhysRevE.74.042102 PACS number(s): 64.60.Ak, 05.60.Cd, 46.65.+g

I. INTRODUCTION

The connectivity of objects in space has many important applications from the spread of diseases and forest fires to the connectivity of fracture networks in rocks used for nuclear waste disposal or for hydrocarbon recovery [2]. Percolation theory is the obvious tool to describe such systems, but they are never infinite or even very large compared with the size of the connecting objects so we must use finite-size scaling to describe the connectivity. However, as well as being finite the shapes or the system, the objects or their orientation is rarely isotropic. For example, in fractured rock, fracture sets with particular orientations are typically found [3,4].

There are few studies of the anisotropic behavior in percolation. Monetti and Albano [5] used an effective threshold, which was defined as the point where the probability to find a percolating cluster is 90% and performed Monte Carlo simulations in an elongated geometry to obtain the dependency of the horizontal and vertical finite-size percolation threshold to the aspect ratio of the lattice. Marrink and Knackstedt [6] have also derived the scaling for the percolation threshold of elongated lattices based on the assumption that an elongated lattice can be treated as a series of linked isotropic lattices. They found that their results deviate from the prediction of Monetti and Albano [5] for aspect ratios greater than four. Langlands *et al.* [7] used linear expansion for the sum of the horizontal and vertical crossing probability to find numerically the dependency of the crossing probability on the aspect ratio of rectangular systems. Hovi and Aharony [8] used the renormalization group theory and duality arguments to treat the correction to the scaling of spanning probability for aspect ratio in rectangular systems. They showed that adding the corrected function is an odd function of a newly defined variable representing the aspect ratio of the system, in line with Cardy's analytical expression [9] derived based on the conformal field theory. Watanabe *et al.* [10] have studied the scaling behavior of the existence probability (the probability that a system percolates) on the two-dimensional rectangular domains with different aspect ratios. They have pointed out that the nonlinearity in the derived scaling is because in rectangular domains the correlation function is not isotropic.

In this paper we extend these ideas and show how effects of anisotropy on the finite-size scaling of connected fraction P can be incorporated. By isotropy we mean that the horizontal connectivity is the same as the vertical connectivity on average if not for individual realizations. For anisotropic systems there will be an *easy* direction for connected paths to be formed and a *difficult* direction. We studied two-dimensional lattices of size $n_x \times n_y$ and define an aspect ratio $\omega = n_x / n_y$. Free boundary conditions in both x and y directions are considered and various clusters are identified using standard algorithms [11]. Then, in the usual way we investigate percolation properties as a function of the occupancy probability p. We fix the aspect ratio and investigate finite-size scaling as the overall system size (now defined by n_x) varies. If $\omega > 1$ then y is the short direction and we expect (and observe) connectivity to develop in that direction first. Clearly there is

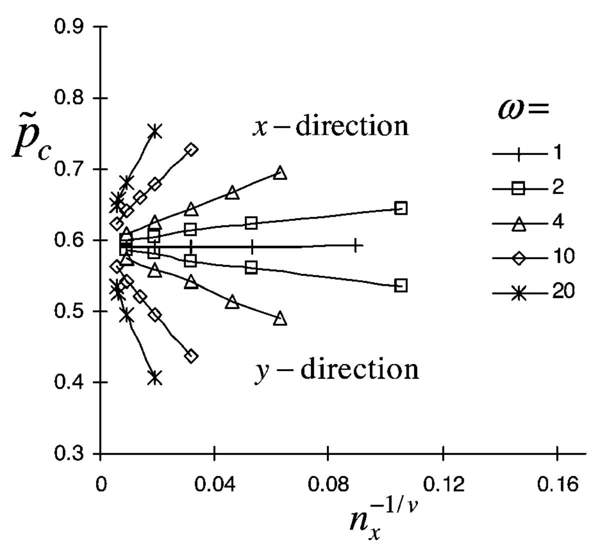

FIG. 1. Plot of apparent threshold in both the x and the y directions as a function of $n_x^{-1/\nu}$, showing that a shift in the apparent thresholds is symmetrically placed about the isotropic case ($\omega = 1$).

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1992).

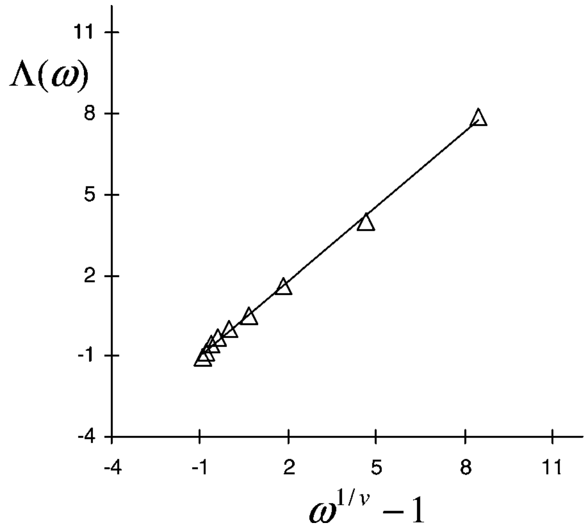

FIG. 2. Plot of the shift in the apparent threshold Λ as a function of aspect ratio ω showing that the proposed scaling function $\Lambda(\omega)=c(\omega^{1/v}-1)$ is satisfied with $c=0.92\pm0.04$.

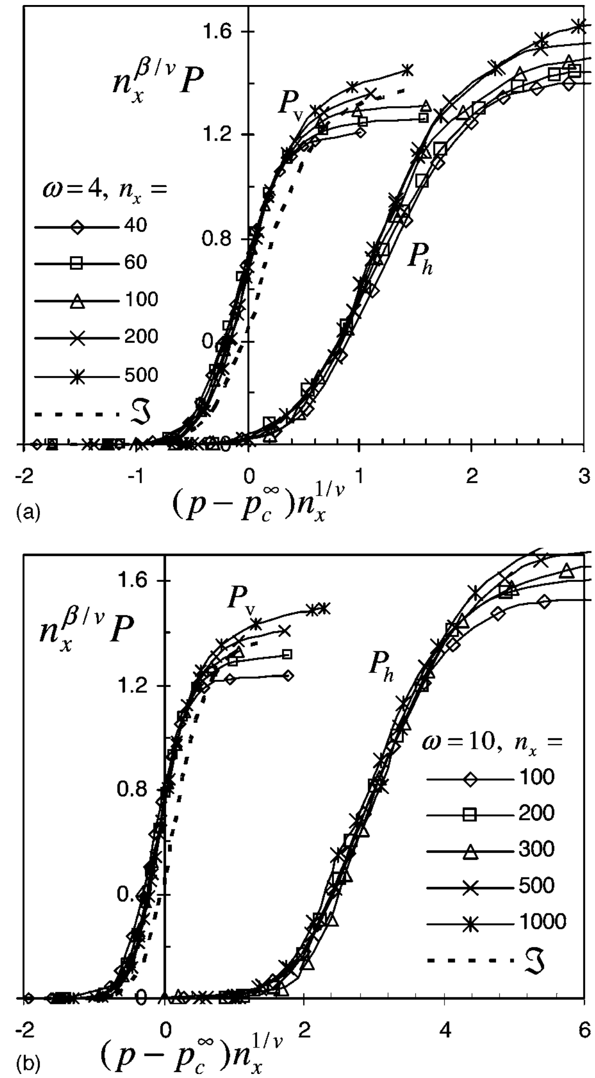

FIG. 3. Rescaled horizontal P_h and vertical P_v mean connectivity curves using infinite threshold for the aspect ratio of (a) $\omega=4$ and (b) $\omega=10$ where $\Im$ is the isotropic mean connectivity curve.

symmetry between connectivity in the x direction for aspect ratio ω and that in the y direction for aspect ratio $1/\omega$. We shall exploit this symmetry later. We only study the impact of a moderate aspect ratio for large (but finite) systems. As the aspect ratio approaches $1/n_y$ the system becomes one dimensional and we would expect to see a crossover to one-dimensional behavior. In that case the universal scaling exponents will be changed. The form of variation of the universal exponents from 2D to 1D values in this regime needs more investigations. For this reason we restrict our attention to aspect ratios less than around ten and to system sizes where the short dimension is greater than around 20. This demonstrates the effects of anisotropy without getting into the regime where lower-dimensional behavior is expected. Here we focus our study to analyze the effects of aspect ratio in anisotropic systems on the connected fraction P and its associated uncertainty Δ. Obviously, there are other interesting aspects that require further research. One such area is to investigate the dependency of the scaling function of backbone or the fractal dimension of percolating cluster on this aspect ratio.

II. APPARENT PERCOLATION THRESHOLD

We first start by examining the impact of aspect ratio on the *apparent* percolation threshold. At a fixed aspect ratio and system size we generate a large number of realizations (200 and 2000 for the largest and smallest system size, respectively). We count the number of realizations that form clusters, which connect from one side of the system in the x and the y directions. This is equivalent to the second spanning rule previously defined by Reynolds *et al.* [12] where the cell percolates if there exists a cluster that spans the lattice in one given direction. The occupancy probability p, which leads to 50% of realizations connecting in the x direction we call the apparent x threshold $\tilde{p}_c^x$, with a similar definition in the y direction. This implies a special significance that the spanning probability of an infinite size system at p_c is equal to $\frac{1}{2}$. This has been shown by using the renormalization-group theory [8] and the conformal mapping [7,9]. As pointed out by Grassberger [13] this is also consistent with the results of bond percolation for which spanning probability of any finite-size system at p_c is $\frac{1}{2}$. We expect both of the apparent x and y thresholds to scale with the system size as

$$\tilde{p}_c^i = p_c^\infty + \Lambda_i(\omega) n_x^{-1/v}, \tag{1}$$

where p_c^∞ is the usual infinite system-size percolation threshold (0.59275 for site percolation [1]) and i labels the coordinate direction x or y. Note that the anisotropy does not impact on the infinite percolation threshold because then the boundaries cannot affect the clusters.

In Fig. 1 we plot the apparent threshold determined in the x and the y directions for five different aspect ratios as a function of $n_x^{-1/v}$. This shows that indeed the above scaling for the apparent threshold is obeyed. We find from these

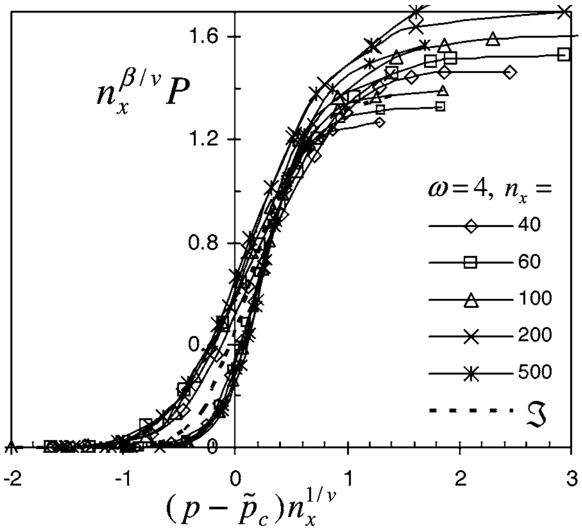

FIG. 4. Data collapse using the finite-size apparent threshold for the aspect ratio of 4 where $\Im$ is the isotropic mean connectivity curve.

results that the separation of the apparent thresholds is symmetrically placed about the isotropic case ($\omega=1$). This implies, at least to a reasonable approximation, that

$$\Lambda_x(\omega) = -\Lambda_y(\omega). \tag{2}$$

This symmetry arises in two dimensions because percolation for the sites in the x direction precludes percolation for the voids in the y direction (and vice versa). Notice that this is not true if one uses periodic boundary condition. This is an approximate symmetry that arises from our choice of the 50% criterion for the apparent threshold. This differs from the definition used by Monetti and Albano [5] who do not see this symmetry. This means that we can define a single function $\Lambda(\omega)$ such that $\tilde{p}_c^x - p_c^\infty = \Lambda(\omega) n_x^{-1/v}$ and $\tilde{p}_c^y - p_c^\infty = -\Lambda(\omega) n_x^{-1/v}$.

Owing to the symmetry mentioned above (swapping x and y) labels we must have that $\tilde{p}_c^x(\omega) = \tilde{p}_c^y(1/\omega)$ or $\Lambda(\omega) n_x^{-1/v} = -\Lambda(1/\omega) n_y^{-1/v}$. This indicates that $\Lambda(\omega)$ has the following property:

$$\Lambda(\omega) = -\omega^{1/v}\Lambda(1/\omega). \tag{3}$$

Neglecting a very small finite-size shift in the threshold for the isotropic case $[\Lambda(\omega=1)\approx 0]$, one simple scaling function that satisfies this is

$$\Lambda(\omega) = c(\omega^{1/v} - 1). \tag{4}$$

In Fig. 2 we plot Λ against $\omega^{1/v}-1$ and find that there is reasonable agreement with this conjecture. Having obtained the aspect ratio dependency of the apparent threshold, we are now able to investigate the finite-size scaling transformations.

III. SCALING OF MEAN CONNECTIVITY

We expect the average connected fraction (P) to follow standard finite-size scaling [1]. In Fig. 3(a) we plot $n_x^{\beta/v}P$ against $(p-p_c^\infty)n_x^{1/v}$ for a fixed value of the aspect ratio of 4. We find that indeed the curves do collapse, however, the vertical and horizontal connectivity curves are separated as we would expect. As the aspect ratio increases this separation increases [Fig. 3(b) for aspect ratio of 10].

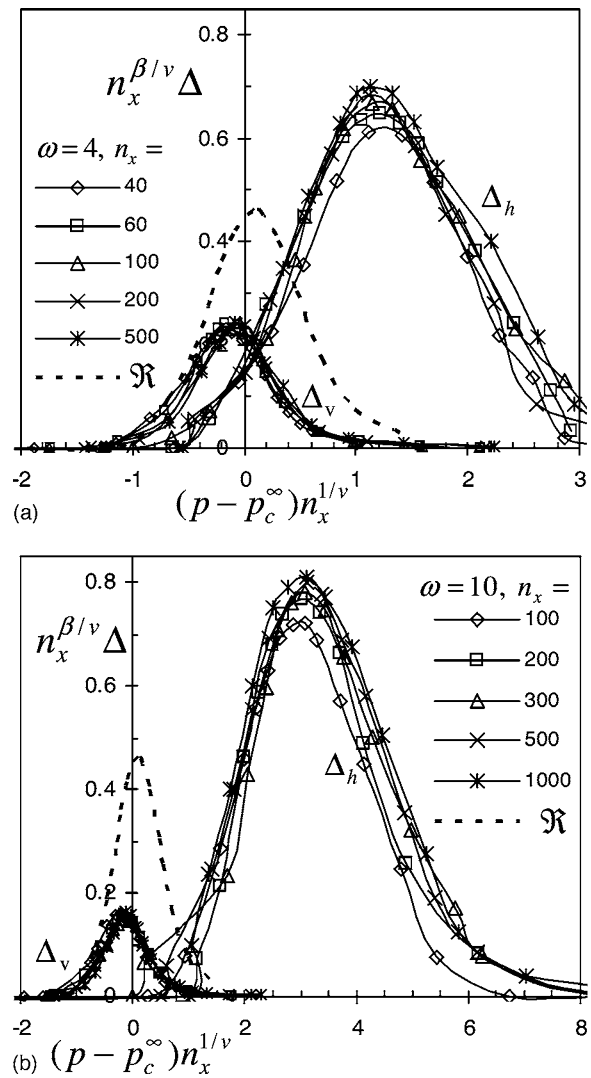

FIG. 5. Rescaled horizontal Δ_h and vertical Δ_v standard deviation of connectivity curves using the infinite threshold and aspect ratio of (a) $\omega=4$ and (b) $\omega=10$ where $\Re$ is the isotropic standard deviation of the connectivity curve.

However, if instead of the infinite percolation threshold we use the finite-size effective threshold as discussed above we find that the curves collapse onto a single curve as shown in Fig. 4.

Similarly, we can rescale the standard deviation of the connectivity $\Delta = \sqrt{(P-\bar{P})^2}$ with the usual finite-size scaling law (Fig. 5). Again the vertical and horizontal connectivity curves are displaced about the isotropic curve. As the aspect ratio increases, both the separation and the magnitudes of the curves increase. To bring the standard deviation of the connectivity curves back to the isotropic curve we have to use the finite-size apparent threshold as discussed above as well as a change in magnitude, which can be accounted for

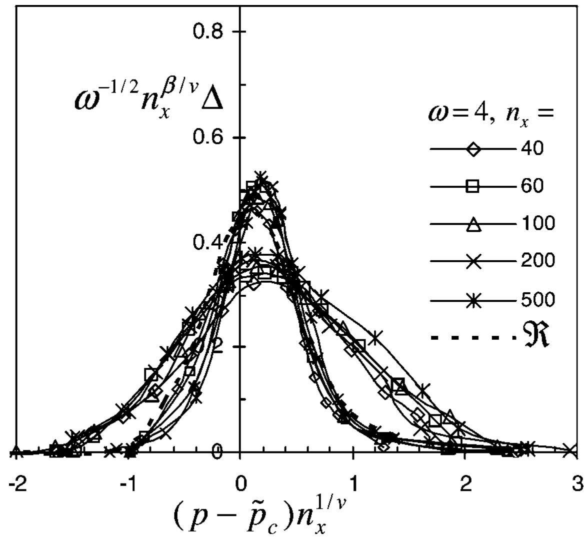

FIG. 6. Data collapse using the finite-size apparent threshold for the aspect ratio of 4 where $\Re$ is the isotropic standard deviation of the connectivity curve.

by rescaling with the geometric mean length, $(n_x n_y)^{1/2}$. From the derivation of the variance in connectivity and its relation to the variance in the cluster size as S/n_x^2 where n_x^2 is the area in 2D isotropic systems, one might use $n_x n_y = \omega^{-1} n_x^2$ to represent the area in anisotropic systems. This introduces a prefactor $\omega^{1/2}$ in the scaling law of the standard deviation of connectivity as

$$\Delta(p, n_x, \omega) = \omega^{1/2} n_x^{-\beta/v} \Re[(p - \tilde{p}_c) n_x^{1/v}]. \tag{5}$$

The results of data collapse are shown in Fig. 6, which indicates that this improves the fit in the standard deviation of connectivity results although it is not perfect. These results enable us to use the same isotropic universal curves ($\Im$ and $\Re$) for predicting the connectivity of anisotropic lattices, which is a good enough approximation for engineering purposes.

IV. CONCLUSIONS

In conclusion, we have shown that we can account for moderate anisotropy in finite-size scaling within percolation by first considering the apparent threshold for connectivity in the principal coordinate directions of the anisotropy. We can then include this within the usual finite-size scaling rules.

ACKNOWLEDGMENTS

The authors would like to acknowledge the financial support received by the Iranian Ministry of Science, Research and Technology, BG Group plc, the UK Department of Trade and Industry (DTI), and Statoil.

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1992).
[2] M. Sahimi, *Applications of Percolation Theory* (Taylor and Francis, London, 1994).
[3] J. Bear, C. F. Tsang, and G. de Marsily, *Flow and Contaminant Transport in Fractured Rock* (Academic Press, San Diego, 1993).
[4] M. Sahimi, *Flow and Transport in Porous Media and Fractured Rock* (VCH, Weinheim, Germany, 1995).
[5] R. A. Monetti and E. V. Albano, Z. Phys. B: Condens. Matter **82**, 129 (1991).
[6] S. J. Marrink and M. A. Knackstedt, J. Phys. A **32**, L461 (1999).
[7] R. P. Langlands, C. Pichet, Ph. Pouliot, and Y. Saint Aubin, J. Stat. Phys. **67**, 553 (1992).
[8] J.-P. Hovi and A. Aharony, Phys. Rev. E **53**, 235 (1996).
[9] J. L. Cardy, J. Phys. A **25**, L201 (1992).
[10] H. Watanabe, S. Yukawa, N. Ito, and C.-K. Hu, Phys. Rev. Lett. **93**, 190601 (2004).
[11] J. Hoshen and R. Kopelman, Phys. Rev. B **14**, 3438 (1976).
[12] P. J. Reynolds, H. E. Stanley, and W. Klein, Phys. Rev. B **21**, 1223 (1980).
[13] P. Grassberger, J. Phys. A **25**, 5475 (1992).